gives rise to a quotient group of prime order as was noted above. If two such subgroups give rise to quotient groups of the same prime order p their cross-cut gives rise to the non-cyclic group of order p^2, and if two such subgroups give rise to quotient groups of different prime orders their cross-cut gives rise to the cyclic group whose order is the product of these two prime numbers. From these facts it follows directly that if all the maximal proper subgroups of G are invariant then G is the direct product of its Sylow subgroups and vice versa. Cf. O. Ore, *Duke Mathematical Journal*, **5**, 431 (1939).

ON FUNCTIONS WHOSE DERIVATIVES DO NOT VANISH IN A GIVEN INTERVAL

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Communicated March 12, 1941

In what follows, J denotes an interval on the real axis and $f(x)$ denotes a function taking real values and possessing all derivatives in J. D. V. Widder proved recently[1] the following interesting theorem.

W. If, for $m = 0, 1, 2, \ldots,$

$$(-1)^m f^{(2m)}(x) \geqq 0$$

in J, *then* $f(x)$ *is an entire function of exponential type.*[2]

Theorem W is closely related to the following result of Serge Bernstein.[3]

B. If no derivative of $f(x)$ *changes sign in* J *and if there exists a fixed* k, *such that it is impossible to find more than* k *successive derivatives of* $f(x)$ *whose absolute values vary all in the same direction (all increase or all decrease), then* $f(x)$ *is an entire function of exponential type.*

Theorem B is of a more general character than theorem W but B does not contain W because B assumes and W does not assume that each derivative of odd order has a constant sign in J. To the problem arising from this situation I should like to contribute the following theorems.

I. *Let* α *be a real number,* $0 < \alpha < \pi$. *If*

$$f^{(n)}(x) \sin (n + 1)\alpha \geqq 0$$

in J *for* $n = 0, 1, 2, 3, \ldots,$ *then* $f(x)$ *is an entire function of exponential type.*

II. *If, for* $m = 0, 1, 2, \ldots,$

$$f^{(4m)}(x) \geqq 0$$

in J and $f^{(4m+1)}(x)$ reaches its maximum in J in the left-hand end-point of J, then $f(x)$ is an entire function of exponential type.

Theorem I contains W as the special case $\alpha = \pi/2$. II contains W because, if $f^{(4m+2)}(x) \leqq 0$ in J, $f^{(4m+1)}(x)$ reaches certainly its maximum in J in the left-hand end-point. But II is evidently not contained in B. Concerning I, we have to distinguish two cases. If α is a rational multiple of π, $\sin\,(n+1)\alpha$ vanishes for an infinity of integers n, and the theorem I makes no hypothesis on the sign of the corresponding $f^{(n)}(x)$ and so this case is not contained in B. If α is not a rational multiple of π, each derivative of $f(x)$ is supposed to keep a constant sign in J, and this case of I is, as a little consideration shows, contained in B.

Outline of the Proofs.—Admit that J is the interval $0 \leqq x \leqq 1$. For both theorems, I and II, I use the formula of partial integration written as follows.

$$\int_0^a S^{(n)}(a-t)F(t)dt = \int_0^a S(a-t)F^{(n)}(t)dt + \sum_{\nu=0}^{n-1}\left(S^{(n-\nu-1)}(a)F^{(\nu)}(0) - S^{(n-\nu-1)}(0)F^{(\nu)}(a)\right) \qquad (1)$$

I. Put in (1)

$$a = \pi, \quad S(t) = e^{t \operatorname{cotg} \alpha} \sin t, \quad F(t) = f(xt/\pi).$$

We can find arbitrarily large integers n such that

$$\frac{n\alpha}{\pi} - \left[\frac{n\alpha}{\pi}\right] < 1 - \frac{\alpha}{\pi},$$

$$\frac{n\alpha}{\pi} - \left[\frac{n\alpha}{\pi}\right] \leqq \frac{(\nu+1)\alpha}{\pi} - \left[\frac{(\nu+1)\alpha}{\pi}\right] \text{ for } \nu = 0, 1, 2, \ldots, n-2.$$

Owing to this choice of n, by the hypothesis of theorem I, all terms on the right hand side of (1) become of the same sign and we obtain an appropriate estimate for the derivatives of $f(x)$.

II. There exists a positive characteristic value a such that the differential equation $y'''' = y$ possesses an integral $y = S(x)$ satisfying the following conditions:

$$S(0) = S'(0) = S(a) = S'(a) = 0,$$

$$S(x) > 0 \text{ for } 0 < x < a.$$

Put this $S(x)$, this a, $n = 4m (m = 1, 2, 3, \ldots)$ and $F(t) = f(xt/a)$ in (1) and an appropriate estimate for the derivatives of $f(x)$ readily follows.

[1] *Proc. Nat. Acad. Sci.*, **26**, 657–659 (1940). A simpler proof was communicated to the New York meeting of the American Mathematical Society, February 22, 1941, by R. P. Boas. Essentially the same proof was found independently by I. J. Schoen-

berg. The idea of this proof is used in the following. I am indebted to Mr. Boas for information about the literature of the subject.

[2] I.e., there exists a positive constant a such that $f(z)e^{-a|z|}$ is bounded for all values of z.

[3] *Comm. Kharkov Math. Soc.*, Series 4, **2**, 1–11 (1928). (See p. 5.)

NOTE ON A CANONICAL FORM FOR THE LINEAR q-DIFFERENCE SYSTEM

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Communicated March 10, 1941

The modern theory of ordinary linear q-difference equations may be considered definitely to have evolved with the appearance of papers by R. D. Carmichael[1] and G. D. Birkhoff[2] announcing what was then regarded as a general theory. Proof of the existence of a full complement of formal solutions in the truly general case and an investigation of their convergence was later given by C. R. Adams.[3] More recently W. J. Trjitzinsky[4] has discovered a full complement of analytic solutions and has developed certain of their asymptotic properties.

As a point of departure either the single linear q-difference equation of the nth order or the equivalent q-difference system may be employed. It is the purpose of this note to announce the existence and indicate the derivation of a canonical form for the linear q-difference system of the nth order analogous to that found by Birkhoff for the linear differential system.[5] Consider the matrix form of the q-difference system

$$Y(qx) = A(x)Y(x), \qquad |q| \neq 1, \tag{1}$$

where $A(x)$ is analytic or has a pole at $x = \infty$, and $|A(x)| \not\equiv 0$. From the work of Adams cited above, such a system (1) has always a full complement of n formal solutions, which in the typical case that certain rational numbers $\mu_1, \mu_2, \ldots, \mu_n$ are all distinct may be written (after a simple preliminary transformation) in the form

$$Y(x) = (s_{ij}(x))(q^{\frac{\mu}{2}(t^2-t)} x^{r_i}\delta_{ij}),$$

where the $s_{ij}(x)$ are power series of the form $\sum_{k=0}^{\infty} s_{ijk}x^{-k}$, $(s_{ij}(\infty)) = (\delta_{ij})$ In this case the matrix $A(x)$ in (1) has the special form